MLS: Joint Manifold-Learning and Sparsity-Aware Framework for Highly Accelerated Dynamic Magnetic Resonance Imaging [1]

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[1] Nakarmi, Ukash, Konstantinos Slavakis, and Leslie Ying. "MLS: Joint manifold-learning and sparsity-aware framework for highly accelerated dynamic magnetic resonance imaging." (ISBI 2018), IEEE, 2018.

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Introduction

- Manifold based models more efficient than conventional low-rank approaches
- Proposed a joint manifold learning and sparsity-aware framework for dynamic MRI
- Method establishes link between recently developed manifold models and conventional sparsity-aware models

Background

Dynamic image series represented by an $N \times N_{fr}$ Casorati Matrix:

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N_{fr}}]$$

where N is the size of each image $(N_p \times N_f)$ and $\mathbf{x}_i$ is the i^{th} image.

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where ϕ is the Fourier undersampling operator and $\mathbf{V}$ stands for noise.

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Recovering image series from undersampled k-space data

$$\arg\min_{\mathbf{X}} ||\mathbf{Y} - \phi(\mathbf{X})||_F^2 + \lambda R(\mathbf{X})$$

where $R(\cdot)$ is the Fourier sparsity-aware loss along temporal diretion.

Algorithm

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3. Reconstruct the dynamic image series using regularized inverse problem framework.

Manifold Learning and Embedding

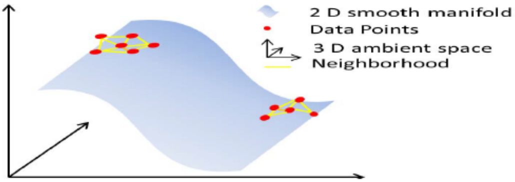

Figure: 3D data points lying close to the 2D smooth manifold surface.

- Assume there exists a smooth M-dimensional $(M \ll N)$ manifold $\mathcal{M} \in \mathcal{C}^N$ such that an image $x_i \in \mathcal{C}^N$ lie on or close to $\mathcal{M}$.

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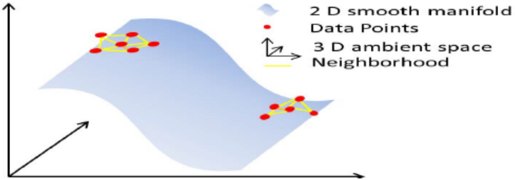

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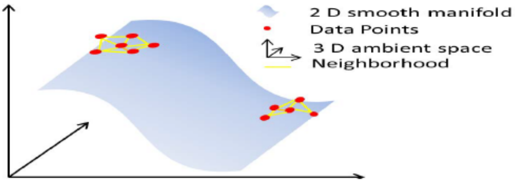

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- Neighborhoods defined in terms of properties of the tangent spaces of a smooth manifold.

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Image $\mathbf{x}_i$ approximated by the affine combination of its neighbors:

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- Constraint $\omega_i^H \mathbf{1}_{N_{fr}} = 1$ ensures affine neighboring relations.
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Result: The Manifold geometry of dynamic image series is described by an $N_{fr} \times N_{fr}$ weight matrix $\mathbf{W}$ whose entries are ω_i^n.

Objective of Manifold Learning

Find an M dimensional basis $\boldsymbol{\Psi}$ that preserves the manifold geometry

$$\underset{\substack{\boldsymbol{\Psi} \in \mathcal{C}^{M \times N_{fr}}, \\ \boldsymbol{\Psi}\boldsymbol{\Psi}^H = \mathbf{I}_M, \boldsymbol{\Psi}\mathbf{1}_M = \mathbf{0}_M}}{\arg\min} \quad \sum_{i=1}^{N_{fr}} ||\psi_i - \sum_{n=1}^{N_{fr}} \omega_i^n \psi_n||^2$$

where $\boldsymbol{\Psi} = [\psi_1 \, \psi_2 \, \cdots \, \psi_{N_{fr}}]$.

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- **Constraint:** 1) $\boldsymbol{\Psi}\boldsymbol{\Psi}^H = \mathbf{I}_M$ excludes all-zero solution and 2) $\boldsymbol{\Psi}\mathbf{1}_M = \mathbf{0}_M$ centers the columns of $\boldsymbol{\Psi}$ around $\mathbf{0}$.
- **Solution:** The desired $\boldsymbol{\Psi}$ is given by the eigen-decomposition of the matrix $\kappa := (I - \mathbf{W})(I - \mathbf{W})^H$ such that rows ψ^m $(m = 1, 2, \cdots, M)$ of $\boldsymbol{\Psi}$ are the eigenvectors κ that correspond to the M least significant eigenvalues.

Dynamic MRI: Reconstruction from undersampled k-space data

- Use $\boldsymbol{\Psi}$ as the temporal basis of dynamic image series
- Represent image series Casorati Matrix as $\mathbf{X} = \mathbf{U}\boldsymbol{\Psi}$

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- Sparsity Enforcing regularizer is given by:

$$R(\mathbf{U}\boldsymbol{\Psi}) = ||U\boldsymbol{\Psi}_f||$$

where $\boldsymbol{\Psi}_f = \mathcal{F}(\boldsymbol{\Psi})$ and $\mathcal{F}$ is the Fourier transform operator (suitable for sparse representation) along temporal direction.

Image Reconstruction

Introducing an auxiliary variable: $\mathbf{Z} = \mathbf{U}\mathbf{\Psi}_f$

$$\underset{\mathbf{U},\mathbf{Z}}{\arg\min} ||\mathbf{Y} - \phi(\mathbf{U}\mathbf{\Psi})||_F^2 + \frac{\lambda}{2\delta}||\mathbf{U}\mathbf{\Psi}_f - \mathbf{Z}||_F^2 + \lambda||\rho(\mathbf{Z})||_1$$

where $\rho(\cdot) : \mathcal{C}^{P\times Q} \rightarrow \mathcal{C}^{D\times 1}$, $D = P \times Q$ is a vectorizing operator.

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Iteratively alternating over U and Z until convergence

$$\mathbf{Z}^{(t)} = \arg\min_{\mathbf{Z}} \frac{1}{2\delta}||\mathbf{U}^{(t-1)}\boldsymbol{\Psi}_f - \mathbf{Z}||_F^2 + ||\rho(\mathbf{Z})||_F^2$$

$$\mathbf{U}^{(t)} = \arg\min_{\mathbf{U}} ||\mathbf{Y} - \phi(\mathbf{U}\boldsymbol{\Psi})||_F^2 + \frac{\lambda}{2\delta}||\mathbf{U}\boldsymbol{\Psi}_f - \mathbf{Z}^t||_F^2$$

Once an optimal $\mathbf{U}^*$ is obtained at convergence, desired dynamic image series can be computed as $\mathbf{X} = \mathbf{U}^*\boldsymbol{\Psi}$.