Time and Accuracy Considerations of Uncompressed vs Compressed QR Based Least Squares Solution

Avrajit Ghosh, Siddhant Gautam

Computational Mathematics, Science and Engineering,

Michigan State University

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- Find the best values of m and n for a given N for a good error-time trade-off

Introduction

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- Find the best values of m and n for a given N for a good error-time trade-off

$$\underbrace{\begin{pmatrix} f_{11} & \cdots & f_{1N} \\ f_{21} & \cdots & f_{2N} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mN} \end{pmatrix}}_{RFD_{m \times N}} \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{Nn} \end{pmatrix}}_{A_{N \times n}} = \underbrace{\begin{pmatrix} ac_{11} & \cdots & ac_{1n} \\ ac_{11} & \cdots & ac_{1n} \\ \vdots & \ddots & \vdots \\ ac_{m1} & \cdots & ac_{mn} \end{pmatrix}}_{Ac_{m \times n}}$$

Gram-Schimdt Orthogonalization

Consider a matrix $\mathbf{A}$

$$\mathbf{A} = [\mathbf{a}_1 | \mathbf{a}_2 | \cdots | \mathbf{a}_n]$$

Then,

$$\mathbf{u}_1 = \mathbf{a}_1, \;\; \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1, \;\; \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$

$$\mathbf{u}_{k+1} = \mathbf{a}_{k+1} - (\mathbf{a}_{k+1} \cdot \mathbf{e}_1)\mathbf{e}_1 - \cdots - (\mathbf{a}_{k+1} \cdot \mathbf{e}_k)\mathbf{e}_k, \;\; \mathbf{e}_{k+1} = \frac{\mathbf{u}_{k+1}}{\|\mathbf{u}_{k+1}\|}$$

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$$\mathbf{A} = \left[\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n\right]_{N\times n} = \left[\mathbf{e}_1|\mathbf{e}_2|\cdots|\mathbf{e}_n\right]_{N\times n} \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{e}_1 & \mathbf{a}_2 \cdot \mathbf{e}_1 \cdots \mathbf{a}_n \cdot \mathbf{e}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{e}_2 \cdots \mathbf{a}_n \cdot \mathbf{e}_2 \\ \vdots \ddots \vdots & \\ 0 & 0 \cdots \mathbf{a}_n \cdot \mathbf{e}_n \end{bmatrix}_{n\times n}$$

Moore-Penrose Pseudo Inverse

Least Squares Solution for Overdetermined System

$$\hat{x} = \arg\min_{x} \|Ax - b\|_2$$

where $A \in \mathbb{C}^{N \times n}$ is a tall matrix i.e. $N > n$ and $b \in \mathbb{C}^N$ is not in the column space of A.

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Standard Approach: QR-decomposition $A = QR$

$$\hat{x} = (A^T A)^{-1} A^T b = (R^T Q^T Q R)^{-1} A^T b = (R^T R)^{-1} A^T b$$

where $R \in \mathbb{C}^{n \times n}$ is upper triangular matrix and multiplying it and inverting the result is fast.

RFD matrix as Subspace Embedding

Theorem-1: Low dimensional Subspace embedding

$$(1-\epsilon)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\epsilon)\|x\|_2^2 \;\; \forall x \in \mathcal{L}_{\mathcal{B}}^r$$

if $\Phi \in \mathbb{C}^{m \times n}$ is an $\epsilon/2$-JL map of C into $\mathbb{C}^m$ and $C \subset S_{\mathcal{B}}^r$ is a minimal $\epsilon/16$-cover of the compact set $S_{\mathcal{B}}^r \subset \mathcal{L}_{\mathcal{B}}^r$.

Compressing Overdetermined System of Equations

$$(1-\epsilon)\|Ay-b\|_2^2 \leq \|\Phi(Ay-b)\|_2^2 \leq (1+\epsilon)\|Ay-b\|_2^2$$

where $Ay - b \in \mathcal{L}_{\mathcal{B}}^{n+1}$ and $\mathcal{B} = \{a_1, \cdots, a_n, b\}$ be the $n+1$ ortho normalized columns of A and b.
Φ can be an $\tilde{F} = \sqrt{N/m}\, RFD$ matrix where R is random row selection matrix, F is the DFT matrix and D is diagonal matrix with Radamacher variables.

Error Consideration of RFD-Compression

Fast JL embedding matrix *RFD* satisfies

$$(1-\epsilon)\|Ay-b\|_2^2 \leq \|\tilde{F}(Ay-b)\|_2^2 \leq (1+\epsilon)\|Ay-b\|_2^2$$

where

- $\tilde{F} = \sqrt{N/m}\, RFD \in \mathbb{C}^{m\times N}$ is the fast JL embedding matrix.
- $m = c_1(n+1)\ln(c_2/\epsilon)\ln^4 N$ is the number of rows of JL matrix $\tilde{F}$.
- N and n are the number of rows and columns of matrix A

Solution to the compressed least squares y'_{min} satisfies:

$$\sqrt{\frac{1-\epsilon}{1+\epsilon}}\|Ay_{min}-b\|_2 \leq \|Ay'_{min}-b\|_2 \leq \sqrt{\frac{1+\epsilon}{1-\epsilon}}\|Ay_{min}-b\|_2$$

where $y'_{min} = \arg\min_{z\in\mathbb{C}^n}\|\tilde{F}Az-\tilde{F}b\|_2$ is the solution to the compressed least squares.

Time consideration of RFD-Compression

Time complexity of operations in RFD compression:

- $RFD \times A \longrightarrow \mathbb{O}(nNlog(N))$
- $RFD \times b \longrightarrow \mathbb{O}(Nlog(N))$
- $\arg\min_z \|RFD\ Az - RFD\ b\|_2 \longrightarrow \mathbb{O}(mn^2) = \mathbb{O}(n^3 log^4(N))$

Comparisons of Time complexity for solving Least Squares Problem:

- Uncompressed QR-based least squares: $\mathbb{O}(n^2 N)$
- RFD compression based least squares: $\mathbb{O}(n^3 log^4(N) + nN \log(N))$
- For time advantage, validity range of n: $\log(N) \leq n \leq N \log^{-4}(N)$

Parameters Chosen

Parameters

- $A = [a_{i,j}]_{i \in [N], j \in [n]}$, $b = [b_j]_{j \in [n]}$ where $[a_{i,j}] \sim U(0,1)$, $[b_j] \sim U(0,1)$.
- To reduce the randomness, 25 trails were taken per iteration.
- CPU: Intel(R) Core(TM) $i9 - 9900K$ CPU @ 3.60 GHz, 64 GB RAM.
- n is almost in the range $[\lceil \log(N) \rceil - \eta, \lceil N log^{-4}(N) \rceil + \eta]$, $\eta \in I$.

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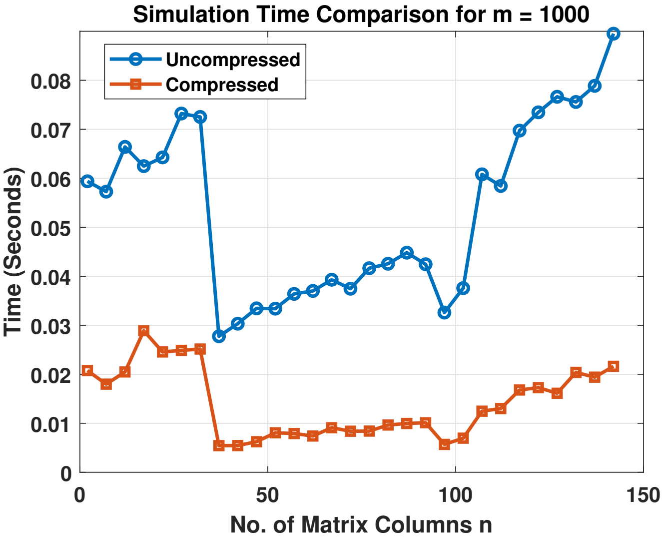
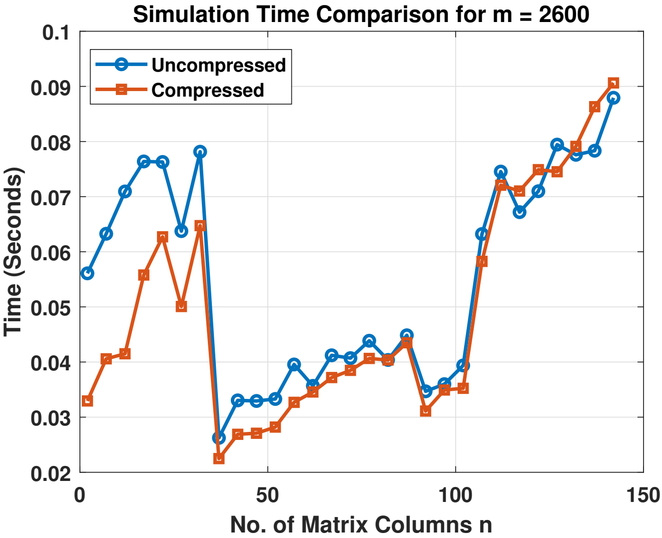
No. of rows N	**Compression Factor m**	**No. of columns n**
1000	400 : 100 : 1000	2 : 5 : 72
3000	1000 : 200 : 3000	2 : 5 : 72
5000	2000 : 500 : 5000	2 : 5 : 142

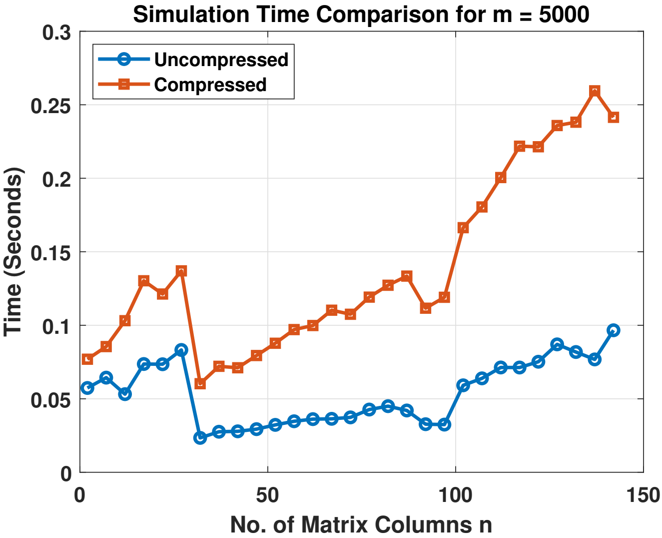
Clear Time advantage for $N = 5000$ and $m = 1000$



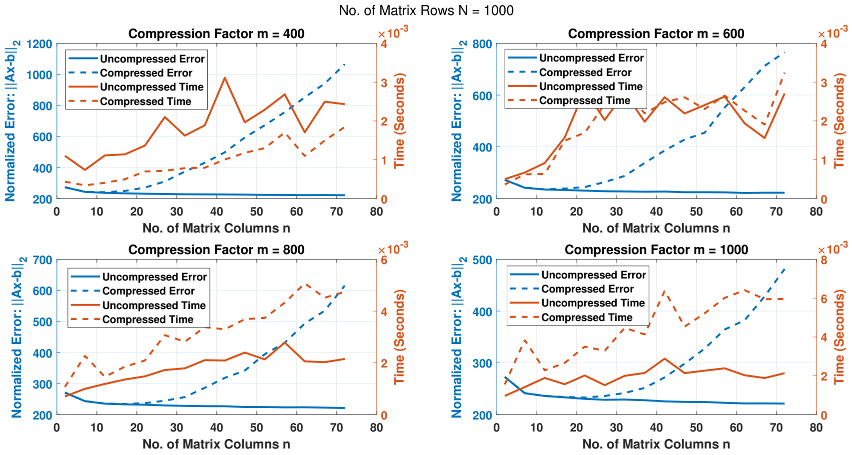
Comparable times for $N = 5000$ and $m = 2600$



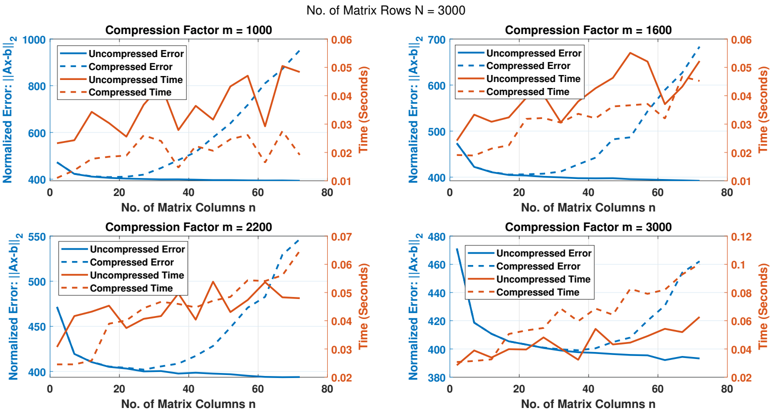
Time disadvantage for $N = 5000$ and $m = 5000$



Error vs Simulation Time for N=1000



Error vs Simulation Time for N=3000



- Trade-off involved between inversion error and simulation time.
- Compressed error always more significant than the uncompressed error.
- Only for low values of m, significant time advantage gained

Selecting range of m and n from time simulations

Upper bound on m

- Time advantage by compression for $m \leq \hat{m}_{ub}$.
- For $m \geq \hat{m}_{ub}$, we will get a disadvantage in time.
- From Observations for $N = 5000$, $\hat{m}_{ub} = 2600$:

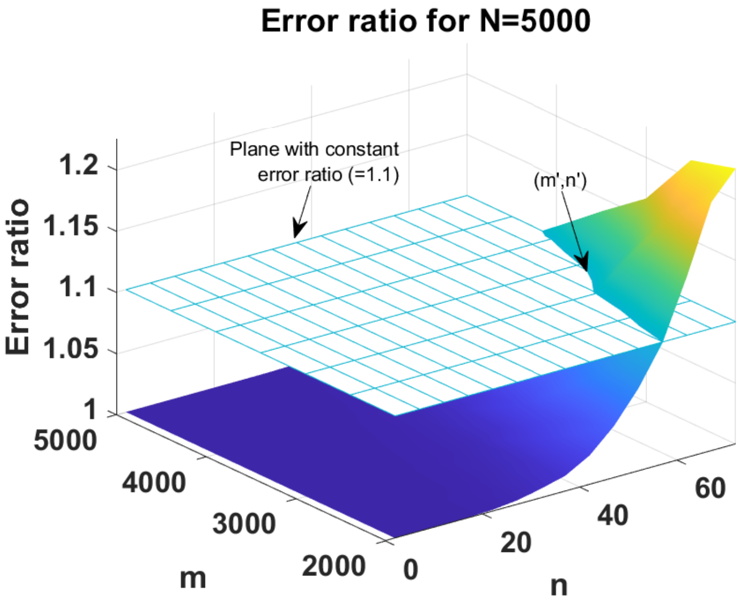
$$mn^2 + nNlogN \leq n^2 N$$

Lower bound on n

$\forall m \leq \hat{m}_{ub} \exists\, n \geq \hat{n}_{lb}(m)$, where $\hat{n}_{lb}(m)$ is the optimum lower bound on n for a given m. Then the maximum lower bound on n will be $\hat{n}_{lb}(\hat{m}_{ub})$: $\hat{n}_{lb} = 10$

$$n \geq \frac{N \log(N)}{N - m}$$

Error Ratio for $N = 5000$



Selecting range of m and n from error simulations

Allowable error threshold

If the error for the compressed least squares lies within 5% of the original least squares error, then

$$\|Ay'_{min} - b\|_2 \leq 1.1\|Ay_{min} - b\|_2 \implies \sqrt{\frac{1+\epsilon}{1-\epsilon}} = 1.1 \implies \epsilon = 0.095$$

Bounds on n and m

- Error ratio decreases with $m \implies$ Lower bound on m: $m \geq \hat{m}_{lb}$
- Error ratio increases with $n \implies$ Upper bound on n: $n \leq \hat{n}_{ub}$

Getting a final bound on n and m

Combining bounds from error and time consideration

- If $\hat{m}_{lb} \leq \hat{m}_{ub}$, a feasible bound for m is $\hat{m}_{lb} \leq m \leq \hat{m}_{ub}$
- If $\hat{n}_{lb} \leq \hat{n}_{ub}$, a feasible bound for m is $\hat{n}_{lb} \leq n \leq \hat{n}_{ub}$

Trade-off between error and time

- Keeping $(\hat{m}_{ub}, \hat{n}_{lb})$ fixed, we want $(\hat{n}_{ub} - \hat{n}_{lb})$ and $(\hat{m}_{ub} - \hat{m}_{lb})$ to increase.
- $\hat{m}_{lb} \downarrow$, , $\hat{n}_{ub} \uparrow$, error ratio $\uparrow$.
- $\hat{n}_{lb} = 7$, $\hat{n}_{ub} = 45$, $\hat{m}_{ub} = 2600$, $\hat{m}_{lb} = 2000$.
- Will get more time-advantage over a large range of m and n, if allowable error ratio increases.

Conclusion

- QR decomposition used in finding least squares solution to overdetermined system of equations.
- Compressed the overdetermined system of equations using RFD matrices.
- Compared the matrix inversion error and time for uncompressed with that of compressed system (RFD).
- Got the lower and upper bounds for compression factor m and the number of columns n for a given number of rows N.
- Trade-off between allowable error and time advantage shown over a range of m and n.

Fin.
Questions?